Root-Finding Absorbing Boundary Condition for Time-Domain Analysis of Wave Propagation and Soil-Structure Interaction

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My Research Topics

- Submerged floating tunnel
- Fixed and floating offshore wind turbine
- Liquid storage tank
- Concrete gravity dam
- Nuclear power plant

Root-Finding Absorbing Boundary Conditions (RFABCs)
Representation of Infinite Media

- The infinite domain is divided into two regions.
- Near-field region
  - Region where the geometry and material properties can be heterogeneous.
  - Nonlinear responses are expected.
  - Usually modeled by conventional nonlinear finite elements.
- Far-field region
  - Infinite region where the geometry is regular and the material properties are homogeneous.
  - Linear responses are assumed.
  - Special mathematical model should be employed which can radiate elastic waves into infinity.

Example of wave-propagation problem in infinite media
Mathematical Models for Infinite Media

- Mathematical models for an infinite region
  - Consistent transmitting boundaries
  - Boundary elements
  - Infinite elements
- Since the above-mentioned approaches are developed in the frequency domain, they are expressed in terms of convolutional operations in the time domain. Although they can produce accurate results for nonlinear wave-propagation analyses in infinite media, their efficiency cannot be ensured because of the convolutional operations.
- To be efficient for the time-domain analyses, the energy radiation into infinity must be represented in terms of local temporal operators in the time domain.
- The high-order absorbing boundary conditions (ABCs) and perfectly matched layers (PMLs) can meet the requirement by adjusting their parameters.
High-order ABCs, PMLs, and new approach: RFABC

- High-order absorbing boundary conditions (ABCs)
  - Approximate the dispersion equation of waves in the infinite region by rational expressions or a series of simple differential operators.
  - Can be implemented easily in the time domain using auxiliary variables.
  - Because of the rational approximation, the effects of computational parameters in the high-order ABCs on their performance can be revealed through mathematical manipulations.

- Perfectly matched layers (PMLs)
  - Introduce artificial damping through complex transformations of the spatial coordinate system.
  - Much easier to implement than the high-order ABCs.
  - Corners in the near-field region can be treated without difficulty.
  - Difficult in practical calculations to determine computational parameters that will guarantee high accuracy.

- In this study, **Root-Finding Absorbing Boundary Conditions (RFABCs)** are newly proposed for scalar- and elastic-waves propagation problems.
  - The new approach is based on solutions of the dispersion equation using a root-finding algorithm such as the Newton-Raphson method.
  - *A new discretization technique* is proposed in order to combine the RFABCs with finite-element models for the near-field region of soil.
Root-Finding Absorbing Boundary Condition for Scalar Waves

- Governing equation & boundary condition
  \[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} \quad \text{in} \quad \Omega_{-\infty} = \{(x, z) \mid x \leq 0, z \in \mathbb{R}\} \]
  \[ B_{\infty} \phi = 0 \quad \text{on} \quad \partial \Omega_{-\infty} = \{(x, z) \mid x = 0, z \in \mathbb{R}\} \]

- Assuming a harmonic motion
  \[ \phi(x, z, t) = \phi(k_x k_z, \omega) \exp[i(\omega t - k_x x - k_z z)] \]

- Exact boundary condition on \( \partial \Omega_{-\infty} \)
  \[ B_{\infty} \phi = \left[ \frac{\partial}{\partial x} + \left( k_x^2 - \frac{\omega^2}{C^2} \right)^{1/2} \right] \phi = \left( \frac{\partial}{\partial x} + S_{\infty} \right) \phi = 0 \]
  \[ S_{\infty} = \left( k_x^2 - \frac{\omega^2}{C^2} \right)^{1/2} \]
Approximation of Dynamic Stiffness

- An approximate expression $\sigma$ for $S_{\infty}$ can be obtained from the following equation.
  \[
f(\sigma) = \sigma^2 - \left( k_z^2 - \frac{\omega^2}{C^2} \right) = 0
  \]

- Applying the Newton-Raphson method
  \[\sigma_0 = \tilde{k}_z + \frac{i\omega}{C} \quad \text{where} \quad \tilde{k}_z \geq 0\]
  \[
  \sigma_n = \frac{1}{2} \left( \sigma_{n-1} + \frac{k_z^2 - \frac{\omega^2}{C^2}}{\sigma_{n-1}} \right) \quad \text{for} \quad n = 1, 2, \ldots
  \]

- Then, approximate boundary condition, RFABC,
  \[B_\infty \phi = \left( \frac{\partial}{\partial x} + S_\infty \right) \phi \approx \left( \frac{\partial}{\partial x} + \sigma_n \right) \phi = 0
  \]
  \[\sigma_0 \phi = \left( \tilde{k}_z + \frac{1}{C} \frac{\partial}{\partial t} \right) \phi
  \]
  \[
  \begin{bmatrix}
  \sigma_n \phi \\
  0
  \end{bmatrix} =
  \begin{bmatrix}
  \frac{1}{2} \sigma_{n-1} & \frac{1}{2} \\
  \frac{1}{2} \left( \frac{\partial^2}{\partial z^2} - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \right) & \frac{1}{2} \sigma_{n-1}
  \end{bmatrix}
  \begin{bmatrix}
  \phi \\
  q_n
  \end{bmatrix}
  \quad \text{or} \quad
  \begin{bmatrix}
  \sigma_n \phi \\
  0
  \end{bmatrix} =
  \begin{bmatrix}
  \frac{1}{2} \sigma_{n-1} & \frac{1}{2} \left( \frac{\partial^2}{\partial z^2} - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \right) \\
  \frac{1}{2} \left( \frac{\partial^2}{\partial z^2} - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \right) & \frac{1}{2} \sigma_{n-1}
  \end{bmatrix}
  \begin{bmatrix}
  \phi \\
  q_n
  \end{bmatrix}
  \quad \text{for} \quad n = 1, 2, \ldots
  \]
Comparison of the boundary conditions

![Graph showing the real and imaginary parts of σ_n/(iω/C) and k_z/(ω/C) for different boundary conditions.]

Root-Finding Absorbing Boundary Conditions (RFABCs)
Relative error of the boundary condition

Root-Finding Absorbing Boundary Conditions (RFABCs)
- Analysis of accuracy

- Consider a solution $\phi(x, z, t) = c_1 \exp[i(\omega t - k_x x - k_z z)] + c_2 \exp[i(\omega t + k_x x - k_z z)]$

- Reflection coefficient

$$R_n = \frac{c_2}{c_1} = \frac{\sigma_n + ik_x}{\sigma_n - ik_x}$$

- The recurrence formula of the Newton-Raphson method

$$\sigma_n = \frac{1}{2} \sigma_{n-1} + \frac{1}{2} \left[ \frac{\frac{\partial^2}{\partial z^2} + i \frac{\partial^2}{\partial t^2}}{\sigma_n} \right] - k_x^2 - \frac{\omega^2}{C^2} \sigma_n = \frac{1}{2} \sigma_n - \frac{1}{2} k_x^2$$

- Then,

$$R_n = -\frac{\sigma_n - ik_x}{\sigma_n + ik_x} = \frac{\frac{1}{2} \sigma_{n-1} + \frac{1}{2} \left( k_x^2 - \frac{\omega^2}{C^2} \right)}{\sigma_n} - \frac{1}{2} \left( \sigma_n - ik_x \right)^2 = \frac{1}{2} \sigma_{n-1} - \frac{1}{2} k_x^2 - \frac{1}{2} k_x^2 + ik_x$$

- For $\sigma_0 = \frac{1}{C} \frac{\partial}{\partial t} + \tilde{k}_z = \frac{i\omega}{C} + \tilde{k}_z$, $|R_0| = \left| \frac{i\omega}{C} + \tilde{k}_z - ik_x \right| \leq 1$

- Therefore, $|R_n| = |R_0|^2 \leq 1$
Reflection coefficients

Root-Finding Absorbing Boundary Conditions (RFABCs)
Analysis of stability at the continuous level

Consider a spatially finite solution which grows exponentially as $t$ increases or radiating waves without incident ones.

$$\phi = \exp[st + \gamma x + ik_z z] \quad \text{where} \quad \text{Re}(s) \geq 0 \quad \text{and} \quad \text{Re}(\gamma) \geq 0$$

Because the solution is unstable, it must not satisfy the newly-developed boundary condition.

$$\frac{\partial \phi}{\partial x} + \sigma_n \phi = (\gamma + \sigma_n) \exp[st + \gamma x + ik_z z] \neq 0$$

From the recurrence formula of the Newton-Raphson method

$$\gamma + \sigma_n = \frac{1}{\prod_{j=0}^{n-1}(2\sigma_j)^{\sigma_j}} (\gamma + \sigma_0)^{\sigma_n}$$

For

$$\sigma_0 = \frac{1}{C} \frac{\partial}{\partial t} + \tilde{k}_z = \frac{i\omega}{C} + \tilde{k}_z, \quad (\gamma + \sigma_0) \phi = \left(\gamma + \frac{s}{C} + \tilde{k}_z \right) \phi \neq 0$$

Thus, $(\gamma + \sigma_n) \phi \neq 0$

Solutions which satisfy the developed boundary condition is stable.
A new discretization technique is proposed in order to combine the RFABC with a finite-element model for the near-field region of soil.

Representation using symmetric and antisymmetric modes

\[
\phi(x,z,t) = \sum_{m=0}^{\infty} \phi_m^s(x,t) \cdot \sqrt{\frac{2 - \delta_m^0}{H}} \cos \lambda_m z + \sum_{m=1}^{\infty} \phi_m^a(x,t) \cdot \sqrt{\frac{2}{H}} \sin \lambda_m z = \sum_{m=-\infty}^{\infty} \phi_m(x,t) g_m(z)
\]

\[
\phi_m(x,t) = \begin{cases} 
\phi_m^s(x,t) & \text{for } m = 0,1,2,\ldots \\
\phi_m^a(x,t) & \text{for } m = -1,-2,\ldots 
\end{cases}
\]

\[
g_m(z) = \begin{cases} 
\sqrt{\frac{2 - \delta_m^0}{H}} \cos \lambda_m z & \text{for } m = 0,1,2,\ldots \\
\sqrt{\frac{2}{H}} \sin \lambda_m z & \text{for } m = -1,-2,\ldots 
\end{cases}
\]

\(\lambda_m\) can be chosen as \(2m\pi/H\) (Fourier series expansion) or can be determined to satisfy boundary conditions on \(z = 0\) and \(H\) (eigenfunction expansion of Sturm-Liouville problem).

- When the artificial boundary meets either a physical boundary or another artificial boundary, boundary conditions for corners must be considered.
- When \(\lambda_m = 2m\pi/H\) (Fourier series expansion), wave fields are assumed periodic and expanded in Fourier series. The RFABC will be applied to each term, Therefore, boundary conditions at corner need not be considered because of the periodically expanded wave fields.
- When \(\lambda_m\) is determined to satisfy boundary conditions on \(z = 0\) and \(H\), boundary conditions at corner are satisfied automatically.
- Continuity condition on the truncated boundary

\[ \phi_m(0, t) = \left( \int_0^H g_m(z)N(0, z)dz \right) \Phi(t) = T_m \Phi(t) \]

- Application of the developed RFABC

\[ \left( \frac{\partial}{\partial x} + \sigma_{m,n} \right) \phi_m(0, t) = 0 \]

- Consistent nodal flux

\[ F(t) = \sum_{m=-\infty}^{\infty} F_m(t) = \left( \sum_{m=-\infty}^{\infty} T_m^T \sigma_{m,n} T_m \right) \Phi(t) \quad \text{where} \quad F_m(t) = -\int_0^H N^T(0, z) \frac{\partial \phi_m(x, t)}{\partial x} \bigg|_{x=0} g_m(z)dz \]

- The stability property of \( \sigma_{m,n} \) at the continuous level is preserved even in the discrete level because the transformation matrix \( T_m \) is independent from the temporal variable \( t \).
Application: Earthquake response analysis of a rigid dam

**Exact solution**

\[ p(0, z, t) = \frac{4 \rho C}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} \cos l_m z \int_0^\infty \tilde{u}_x (\tau) J_0[l_m C(\tau-t)] d\tau \]

- **Root-Finding Absorbing Boundary Conditions (RFABCs)**
Root-Finding Absorbing Boundary Condition for Elastic Waves

- **Governing equations & boundary conditions**

  \[(\lambda + 2\mu)\nabla \cdot u - \mu \nabla \times \nabla \times u = \rho \ddot{u} \text{ in } \Omega_{-\infty} = \{(x, z) | x \leq 0, z \in R\}\]

  \[\sigma = S_\infty u \text{ on } \partial \Omega_{-\infty} = \{(x, z) | x = 0, z \in R\}\]

- **Using potentials**

  \[u(x, z, t) = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}\]

  \[w(x, z, t) = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}\]

  \[\sigma_x(x, z, t) = \rho \frac{\partial^2 \phi}{\partial t^2} - 2\mu \frac{\partial w}{\partial z}\]

  \[\tau_{xz}(x, z, t) = \rho \frac{\partial^2 \psi}{\partial t^2} + 2\mu \frac{\partial u}{\partial z}\]

- **Scalar-wave equations for the potentials**

  \[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{C_p^2} \frac{\partial^2 \phi}{\partial t^2}\]

  \[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{C_s^2} \frac{\partial^2 \psi}{\partial t^2}\]
Assuming a harmonic potentials
\[ \phi(x, z, t) = \phi(k_{xp}, k_z, \omega) \exp[i(\omega t - k_{xp}x - k_z z)] \]
\[ \psi(x, z, t) = \psi(k_{xs}, k_z, \omega) \exp[i(\omega t - k_{xs}x - k_z z)] \]

Boundary conditions for the potentials on \( \partial \Omega_{-\infty} \)
\[ \left[ \frac{\partial}{\partial x} + \left( \frac{k_z^2 - \omega^2}{C_p^2} \right)^{1/2} \right] \phi \approx \left( \frac{\partial}{\partial x} + \sigma^p_n \right) \phi = 0 \]
\[ \left[ \frac{\partial}{\partial x} + \left( \frac{k_z^2 - \omega^2}{C_s^2} \right)^{1/2} \right] \psi \approx \left( \frac{\partial}{\partial x} + \sigma^s_n \right) \psi = 0 \]

Desired boundary condition on \( \partial \Omega_{-\infty} \)
\[
\begin{bmatrix}
\sigma_x \\
\tau_{xz} \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 & -2\mu & \rho \frac{\partial^2}{\partial t^2} & 0 \\
2\mu & 0 & \rho \frac{\partial^2}{\partial t^2} & 0 \\
0 & 1 & 0 & -\frac{\partial}{\partial x} \\
0 & 1 & -\frac{\partial}{\partial x} & 0
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
\phi \\
\psi
\end{bmatrix}
\approx
\begin{bmatrix}
0 & -2\mu & \rho \frac{\partial^2}{\partial t^2} & 0 \\
2\mu & 0 & \rho \frac{\partial^2}{\partial t^2} & 0 \\
0 & 1 & \sigma^p_n & \frac{\partial}{\partial z} \\
0 & 1 & -\frac{\partial}{\partial z} & \sigma^s_n
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
\phi \\
\psi
\end{bmatrix}
\]
Dynamic stiffness of a half-space

- Root-Finding Absorbing Boundary Conditions (RFABCs)
Reflection coefficients

Root-Finding Absorbing Boundary Conditions (RFABCs)
- Representation of potentials using symmetric and antisymmetric modes

\[ \phi(x, z, t) = \sum_{m=-\infty}^{\infty} \phi_m(x, t) g_m(z) \]

\[ \phi_m(x, t) = \begin{cases} \phi^s_m(x, t) & \text{for } m = 0, 1, 2, \ldots \\ \phi^a_m(x, t) & \text{for } m = -1, -2, \ldots \end{cases} \]

\[ g_m(z) = \begin{cases} \frac{2 - \delta_{m0}}{H} \cos \lambda_m z & \text{for } m = 0, 1, 2, \ldots \\ \frac{2}{H} \sin \lambda_m z & \text{for } m = -1, -2, \ldots \end{cases} \]

- \[ \psi(x, z, t) = \sum_{m=-\infty}^{\infty} \psi_m(x, t) h_m(z) \]

\[ \psi_m(x, t) = \begin{cases} \psi^s_m(x, t) & \text{for } m = 1, 2, \ldots \\ \psi^a_m(x, t) & \text{for } m = 0, -1, -2, \ldots \end{cases} \]

\[ h_m(z) = \begin{cases} \frac{2}{H} \sin \lambda_m z & \text{for } m = 1, 2, \ldots \\ \frac{2 - \delta_{m0}}{H} \cos \lambda_m z & \text{for } m = 0, -1, -2, \ldots \end{cases} \]

- Representation of displacements and stresses

\[ u_x(x, z, t) = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} = \sum_{m=-\infty}^{\infty} \left( \frac{\partial \phi_m}{\partial x} + \lambda_m \psi_m \right) g_m(z) = \sum_{m=-\infty}^{\infty} u_{x,m}(x, t) g_m(z) \]

\[ u_z(x, z, t) = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} = \sum_{m=-\infty}^{\infty} \left( \lambda_m \phi_m + \frac{\partial \psi_m}{\partial x} \right) h_m(z) = \sum_{m=-\infty}^{\infty} u_{z,m}(x, t) h_m(z) \]

\[ \sigma_x(x, z, t) = \rho \frac{\partial^2 \phi}{\partial t^2} - 2 \mu \frac{\partial u_z}{\partial z} = \sum_{m=-\infty}^{\infty} \left( \rho \frac{\partial^2 \phi_m}{\partial t^2} + 2 \mu \lambda_m u_{z,m} \right) g_m(z) = \sum_{m=-\infty}^{\infty} \sigma_{x,m}(x, t) g_m(z) \]

\[ \tau_{xz}(x, z, t) = \rho \frac{\partial^2 \psi}{\partial t^2} + 2 \mu \frac{\partial u_x}{\partial z} = \sum_{m=-\infty}^{\infty} \left( \rho \frac{\partial^2 \psi_m}{\partial t^2} + 2 \mu \lambda_m u_{x,m} \right) h_m(z) = \sum_{m=-\infty}^{\infty} \tau_{xz,m}(x, t) h_m(z) \]
Compatibility conditions on the truncated boundary

\[ u_{x,m}(0,t) = \left( \int_0^H g_m(z)N(0,z)dz \right) \Phi(t) = T_{x,m}U_x(t) \]

\[ u_{z,m}(0,t) = \left( \int_0^H h_m(z)N(0,z)dz \right) \Phi(t) = T_{z,m}U_z(t) \]

Application of the developed RFABC

\[
\begin{bmatrix}
\sigma_{x,m}(0,t) \\
\tau_{zx,m}(0,t) \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 & 2\mu\lambda_m & \rho \frac{\partial^2}{\partial t^2} & 0 \\
2\mu\lambda_m & 0 & 0 & \rho \frac{\partial^2}{\partial t^2} \\
0 & 1 & -\lambda_m & \sigma_{m,n}^s \\
0 & 1 & -\lambda_m & \sigma_{m,n}^s
\end{bmatrix} \begin{bmatrix}
u_{x,m}(0,t) \\
u_{z,m}(0,t) \\
\phi_m(0,t) \\
\psi_m(0,t)
\end{bmatrix}
\]

Consistent nodal forces

\[
\begin{bmatrix}
F_{x,m}(t) \\
F_{z,m}(t) \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\sum_{n=0}^{\infty} F_{x,m,n}(t) \\
\sum_{n=0}^{\infty} F_{z,m,n}(t) \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 & -2\mu T_{x,m}^T \lambda_m T_{x,m} & -\rho T_{x,m}^T \frac{\partial^2}{\partial t^2} & 0 \\
-2\mu T_{z,m}^T \lambda_m T_{z,m} & 0 & 0 & -\rho T_{z,m}^T \frac{\partial^2}{\partial t^2} \\
T_{x,m} & 0 & -\lambda_m & \sigma_{m,n}^s \\
T_{z,m} & 0 & -\lambda_m & \sigma_{m,n}^s
\end{bmatrix} \begin{bmatrix}
U_x(t) \\
U_z(t) \\
\varphi_m(0,t) \\
\psi_m(0,t)
\end{bmatrix}
\]

where

\[ F_{x,m}(t) = -\int_0^H N^T(0,z) \sigma_{x,m}(0,t)g_m(z)dz \]

\[ F_{z,m}(t) = -\int_0^H N^T(0,z) \tau_{zx,m}(0,t)h_m(z)dz \]
Application: Gaussian pulse propagation

Free surface

Gaussian pulse as initial conditions

\[ u_r(x, z, t = 0) = x \exp \left[ -100 \left( x^2 + (z - 2.5)^2 \right) \right] \]

\[ u_z(x, z, t = 0) = z \exp \left[ -100 \left( x^2 + (z - 2.5)^2 \right) \right] \]

Root-Finding Absorbing Boundary Conditions (RFABCs)
Conclusion

- In this study, *Root-Finding Absorbing Boundary Conditions (RFABCs)* were newly proposed for scalar- and elastic-wave propagation problems in infinite domains.

- The new approach is based on *solutions of the dispersion equation using a root-finding algorithm* such as the Newton-Raphson method.

- *A new discretization technique* was proposed in order to combine the RFABCs with finite-element models for the near-field region of soil.

- The developed boundary conditions were applied to scalar- and elastic-wave propagation problems in waveguides.
  - The accuracy of the proposed boundary conditions were verified.
  - The stability of the proposed numerical approach was examined by calculating dynamic responses for 500,000 steps of numerical calculations. No instability was observed in the results.
The stability of existing high-order ABCs and PMLs, which are the most popular approaches for wave-propagation problems, has not been proved completely.

The stability of existing high-order ABCs for elastic waves in waveguides was verified only assuming periodic boundary conditions.


The PMLs for elastic waves in waveguides support temporally growing modes.


It is expected that existing high-order ABCs can produce stable results for problems of elastic waves in waveguides when they are discretized using the newly-proposed technique.

Future plans

- Application to problems in a layered half-space
- Application to nonlinear soil-structure interaction analysis
- Application to poroelastic-wave propagation problems
- Application to wave propagation problems in anisotropic media
Thank you for your attention.

Any question?